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On Jacobi polynomials $(\mathscr{P}_k^{(\alpha,\beta)} : \alpha, \beta > -1)$ and Maclaurin spectral functions on rank one symmetric spaces

Richard Olu Awonusika[1] · **Ali Taheri**[1]

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Abstract The Maclaurin spectral functions associated with the development of the heat kernel on compact rank one symmetric spaces are analysed. Relations with various invariants most notably the heat trace, the Minakshisundaram–Pleijel heat coefficients and the spectral residues are carefully examined and a precise formulation as well as asymptotics ($t \searrow 0$) in terms of the celebrated Jacobi theta functions is represented. A natural class of polynomials and power series encoding structural properties of the heat kernel are introduced and further studied.

Keywords Maclaurin spectral functions · Minakshisundaram–Pleijel heat coefficients · Heat trace · Jacobi polynomials · Symmetric spaces

Mathematics Subject Classification 33C05 · 33C45 · 35A08 · 35C05 · 35C10 · 35C15

1 Introduction

Suppose $(\mathscr{X}, g)$ is a compact N-dimensional ($N \geq 1$) Riemannian manifold without boundary and let $\Delta = \Delta_{\mathscr{X}}$ denote the (*positive*) Laplace–Beltrami operator on $\mathscr{X}$ acting on smooth functions $f \in \mathbf{C}^{\infty}(\mathscr{X})$ and given in local coordinates by

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$$\Delta_{\mathscr{X}} f = -\frac{1}{\sqrt{\det g}} \sum_{j=1}^{N} \partial_j \left(\sum_{k=1}^{N} \sqrt{\det g}\, g^{jk} \partial_k \right) f. \tag{1.1}$$

By basic spectral theory there exists a complete orthonormal basis $(\varphi_k : k \geq 0)$ consisting of eigenfunctions of $\Delta_{\mathscr{X}}$, in the Hilbert space $L^2(\mathscr{X})$, with associated eigenvalues $(\lambda_k : k \geq 0)$ satisfying $\Delta_{\mathscr{X}} \varphi_k = \lambda_k \varphi_k$. Each λ_k has finite multiplicity and the spectrum can be arranged in ascending order $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ while $\lambda_j \nearrow \infty$. Furthermore by orthogonality $(\varphi_j, \varphi_k)_{L^2(\mathscr{X})} = 0$ for $0 \leq j \neq k$ whilst $||\varphi_j||_{L^2(\mathscr{X})} = 1$ for all $j \geq 0$.

The heat semigroup $\left(U(t) = e^{-t\Delta_{\mathscr{X}}} : t > 0\right)$ defined in the usual way admits an integral kernel $K = K(t, x, y)$ that for $t > 0$ and $x, y \in \mathscr{X}$ can be expressed by the spectral sum[1]

$$K(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \varphi_k(x) \varphi_k(y). \tag{1.2}$$

It is easily seen that K is real, symmetric in x and y, i.e., $K(t, x, y) = K(t, y, x)$ and smooth; indeed $K \in \mathbf{C}^\infty((0, \infty) \times \mathscr{X} \times \mathscr{X})$. When $\mathscr{X}$ is a compact rank one symmetric space (see Tables 1–3 below) using the addition formula for the matrix coefficients the heat kernel can be shown to have the form $K(t, x, y) = K_{\mathscr{X}}(t, \theta)$ with

$$K_{\mathscr{X}}(t, \theta) = \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} \Phi_k^{\mathscr{X}}(\theta) e^{-\lambda_k^n(\mathscr{X}) t}. \tag{1.3}$$

Here $\lambda_k^n(\mathscr{X})$ (with $k \geq 0$) are the numerically distinct eigenvalues of $\Delta_{\mathscr{X}}$, $M_k^n(\mathscr{X})$ is the dimension of the eigenspace associated with λ_k^n (i.e., the multiplicity of the eigenvalue λ_k^n), $\Phi_k^{\mathscr{X}}(\theta)$ is the spherical function on $\mathscr{X}$ associated with the eigenvalue λ_k^n, θ is the geodesic distance between the points $x, y \in \mathscr{X}$ and $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$. Remarkably in this setting the spherical functions $\Phi_k^{\mathscr{X}}$ can be expressed in terms of the celebrated Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}$ (with $k \geq 0$) and for suitable choice of parameters $\alpha, \beta > -1$ (see Table 1).

Examples of compact rank one symmetric spaces include the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, the real projective space $\mathbf{P}^n(\mathbb{R}) = \mathbb{S}^n\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, the complex projective space $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$ (of real dimension $2n$), the quaternionic projective space $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$ (of real dimension $4n$) and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension 16) (see, e.g., [35] and [6–8, 36]). These spaces with the exception of $\mathbf{P}^n(\mathbb{R})$ $(n \geq 1)$ and $\mathbb{S}^1$ are all simply connected. Although the cases of particular interest in this paper are when $\mathscr{X}$ is either the sphere $\mathbb{S}^n$, the real projective space $\mathbf{P}^n(\mathbb{R})$ or the complex projective space $\mathbf{P}^n(\mathbb{C})$ most of the results and discussions are kept in the general level.

[1] For heat kernels in Riemannian geometry and applications see the monographs Berger et al. [5], Chavel [10] and Li [22] and the references therein. See also Bakry et al. [4].

Table 1 The parameters a, b, N, α, β associated with the compact rank one symmetric space $\mathscr{X}$

$\mathscr{X}$	a	b	N	α	β
$\mathbb{S}^n$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$
$\mathbf{P}^n(\mathbb{R})$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$
$\mathbf{P}^n(\mathbb{C})$	1	$2(n-1)$	$2n$	$n-1$	0
$\mathbf{P}^n(\mathbb{H})$	3	$4(n-1)$	$4n$	$2n-1$	1
$\mathbf{P}^2(\mathrm{Cay})$	7	8	16	7	3

To proceed let us recall some of the most relevant geometric and spectral data associated with these symmetric spaces that will be needed later on. Indeed these are, the radial Laplacian,

$$\Delta_{\mathscr{X}} = -\frac{\partial^2}{\partial\theta^2} - (a\cot\theta + (1/2)b\cot(\theta/2))\frac{\partial}{\partial\theta}; \tag{1.4}$$

the multiplicity (with $k \geq 0$)

$$M_k^n(\mathscr{X}) = \frac{2(k+\varrho)\Gamma(k+2\varrho)\Gamma((a+1)/2)\Gamma(k+N/2)}{k!\Gamma(2\varrho+1)\Gamma(N/2)\Gamma(k+(a+1)/2)}, \quad \varrho = (a+b/2)/2, \quad N = a+b+1, \tag{1.5}$$

of the eigenvalue $\lambda_k^n(\mathscr{X}) = \lambda_k^{(\alpha,\beta)} = (\varrho+k)^2 - \varrho^2$ ($k \geq 0$) of $\Delta_{\mathscr{X}}$; in the simply connected case; and again the volume

$$\mathrm{Vol}(\mathscr{X}) = 2^N \pi^{\frac{N}{2}} \frac{\Gamma((a+1)/2)}{\Gamma((N+a+1)/2)}. \tag{1.6}$$

Table 1 illustrates the parameters a, b, N, α and β for the symmetric spaces just listed. Note that here $\alpha = (N-2)/2$ and $\beta = (a-1)/2$.

In a similar way Table 2 illustrates the geometric and spectral data stated in (1.4)–(1.6) for each of the compact rank one symmetric spaces above.

For the corresponding data for $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$ see Table 3. (See also [16, 17] and [34–36] for further reference and background on Lie groups and symmetric spaces.) Now referring to the contents of Tables 1, 2 and 3 we can formulate the heat kernels on $\mathscr{X}$.

Table 2 The compact rank one symmetric spaces $\mathscr{X}$

$\mathscr{X}$	$\mathbb{S}^n$	$\mathbf{P}^n(\mathbb{R})$	$\mathbf{P}^n(\mathbb{C})$
$\Delta_{\mathscr{X}}$	$-\frac{\partial^2}{\partial\theta^2} - (n-1)\cot\theta\frac{\partial}{\partial\theta}$	$-\frac{\partial^2}{\partial\theta^2} - (n-1)\cot\theta\frac{\partial}{\partial\theta}$	$-\frac{\partial^2}{\partial\theta^2} - \left(\cot\theta + (n-1)\cot\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}$
$\lambda_k^n(\mathscr{X})$	$k(k+n-1)$	$2k(2k+n-1)$	$k(k+n)$
$M_k^n(\mathscr{X})$	$(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$	$(4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2$
$\mathrm{Vol}(\mathscr{X})$	$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$	$\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$	$\frac{4^n\pi^n}{n!}$

Table 3 The symmetric spaces $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$

$\mathscr{X}$	$\mathbf{P}^n(\mathbb{H})$	$\mathbf{P}^2(\mathrm{Cay})$
$\Delta_{\mathscr{X}}$	$-\frac{\partial^2}{\partial\theta^2}-\left(3\cot\theta+2(n-1)\cot\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}$	$-\frac{\partial^2}{\partial\theta^2}-\left(7\cot\theta+4\cot\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}$
$\lambda_k^n(\mathscr{X})$	$k(k+2n+1)$	$k(k+11)$
$M_k^n(\mathscr{X})$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$
$\mathrm{Vol}(\mathscr{X})$	$\frac{(4\pi)^{2n}}{\Gamma(2n+2)}$	$\frac{3!}{11!}(4\pi)^8$

The Sphere $\mathscr{X}=\mathbb{S}^n$ The sphere $\mathbb{S}^n$ is a compact rank one symmetric space in view of the identification $\mathbb{S}^n=\mathbf{SO}(n+1)/\mathbf{SO}(n)$ where $\mathbf{SO}(n)$ is the special orthogonal group of real $n\times n$ matrices. We assume $\mathbb{S}^n$ is equipped with its canonical metric of sectional curvature $+1$. Using (1.3) and the above data the heat kernel takes the explicit form

$$\begin{aligned} K_{\mathbb{S}^n}(t,\theta) &= \frac{1}{\omega_1^n}\sum_{k=0}^{\infty} M_k^n(\mathbb{S}^n)\Phi_k^{\mathbb{S}^n}(\theta)e^{-k(k+n-1)t} \\ &= \frac{1}{\omega_1^n}\sum_{k=0}^{\infty}(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}\mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta)e^{-k(k+n-1)t}, \end{aligned} \tag{1.7}$$

where $\omega_1^n=\mathrm{Vol}(\mathbb{S}^n)$ (*cf.* Table 2). The spherical or zonal functions $\Phi_k^{\mathbb{S}^n}$ which are given in terms of the Gegenbauer polynomials C_k^{ν} (with $\nu=(n-1)/2$) can be expressed as

$$\Phi_k^{\mathbb{S}^n}(\theta):=\mathscr{C}_k^{\frac{n-1}{2}}(\cos\theta)=\frac{C_k^{\frac{n-1}{2}}(\cos\theta)}{C_k^{\frac{n-1}{2}}(1)},\quad \Phi_k^{\mathbb{S}^n}(0)=1,\quad C_k^{\frac{n-1}{2}}(1)=\frac{\Gamma(k+n-1)}{\Gamma(n-1)k!}, \tag{1.8}$$

which are in turn eigenfunctions of the Laplacian Δ on the sphere $\mathbb{S}^n$, i.e., $\Delta\Phi_k^{\mathbb{S}^n}=k(k+n-1)\Phi_k^{\mathbb{S}^n}$. Note that in obtaining the spectral sum (1.7) we have set $t=\cos\theta$ in the spherical Laplacian $\Delta_{\mathbb{S}^n}$ and consequently the associated spherical eigenvalue equation reduces to the Gegenbauer differential equation (1.9) [with $\nu=(n-1)/2$]

$$\left(1-t^2\right)\frac{d^2y}{dt^2}-(2\nu+1)t\frac{dy}{dt}+k(k+2\nu)y=0, \tag{1.9}$$

with the truncated power series solution $y=C_k^{\frac{n-1}{2}}(t)$. Furthermore, the heat trace formula on $\mathbb{S}^n$ is given by the spectral sum

$$\operatorname{tr} e^{-t\Delta_{\mathbb{S}^n}} = \omega_1^n K_{\mathbb{S}^n}(t,0) = \sum_{k=0}^{\infty}(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}e^{-k(k+n-1)t}, \qquad t>0. \tag{1.10}$$

The Real Projective Space $\mathscr{X} = \mathbf{P}^n(\mathbb{R})$ The real projective space $\mathbf{P}^n(\mathbb{R})$ is a compact rank one symmetric space in view of the identification $\mathbf{P}^n(\mathbb{R}) = \mathbb{S}^n/\{\pm\} = \mathbf{SO}(n+1)/\mathbf{O}(n)$, where $\mathbf{O}(n)$ denotes the orthogonal groups of real $n \times n$ matrices. In view of the double covering description above and using the theory of Riemannian coverings it is not difficult to see that the eigenfunctions of $\mathbf{P}^n(\mathbb{R})$ are exactly those descending from the cover $\mathbb{S}^n$ whose degree of homogeneity is even. As a consequence the numerically distinct eigenvalues of Δ on $\mathbf{P}^n(\mathbb{R})$ are given by $\lambda_k^n = 2k(2k+n-1)$, $k \geq 0$, with the multiplicity $M_k^n(\mathbf{P}^n(\mathbb{R})) = M_{2k}^n(\mathbb{S}^n)$ (see Table 2). Now since $\mathbf{P}^n(\mathbb{R})$ is obtained from $\mathbb{S}^n$ by identifying the antipodal points, its diameter is $\pi/2$ and its volume is $\omega_2^n = \mathrm{Vol}(\mathbf{P}^n(\mathbb{R})) = \omega_1^n/2$ (as shown in Table 2). Moreover the radial Laplacian on $\mathbf{P}^n(\mathbb{R})$ coincides with that on $\mathbb{S}^n$ and subsequently

$$K_{\mathbf{P}^n(\mathbb{R})}(t,\theta) = K_{\mathbb{S}^n}(t,\theta) + K_{\mathbb{S}^n}(t,\pi-\theta). \tag{1.11}$$

Noting that

$$\Phi_k^{\mathbb{S}^n}(\pi-\theta) = \mathscr{C}_k^{\frac{n-1}{2}}(\cos(\pi-\theta)) = \mathscr{C}_k^{\frac{n-1}{2}}(-\cos\theta),$$

and using the identity $C_k^\nu(-t) = (-1)^k C_k^\nu(t)$ on the Gegenbauer polynomials we have

$$\Phi_k^{\mathbb{S}^n}(\pi-\theta) = (-1)^k \Phi_k^{\mathbb{S}^n}(\theta).$$

As a result from (1.11) we therefore obtain

$$\begin{aligned} K_{\mathbf{P}^n(\mathbb{R})}(t,\theta) &= \frac{1}{\omega_1^n}\sum_{k=0}^{\infty} M_k(\mathbb{S}^n)[1+(-1)^k]\Phi_k^{\mathbb{S}^n}(\theta)e^{-k(k+n-1)t} = \frac{e^{\frac{(n-1)^2}{4}t}}{\omega_2^n}\sum_{k=0}^{\infty} M_{2k}(\mathbb{S}^n) \\ &\quad \times \Phi_{2k}^{\mathbb{S}^n}(\theta)e^{-\left(2k+\frac{n-1}{2}\right)^2 t} = \frac{e^{\frac{(n-1)^2}{4}t}}{\omega_2^n}\sum_{k=0}^{\infty} M_{2k}(\mathbb{S}^n)\mathscr{C}_{2k}^{\frac{n-1}{2}}(\cos\theta)e^{-\left(2k+\frac{n-1}{2}\right)^2 t}, \end{aligned} \tag{1.12}$$

and as a consequence the heat trace on $\mathscr{X} = \mathbf{P}^n(\mathbb{R})$ takes the form ($t > 0$)

$$\operatorname{tr} e^{-t\Delta_{\mathbf{P}^n(\mathbb{R})}} = \omega_2^n K_{\mathbf{P}^n(\mathbb{R})}(t,0) = \sum_{k=0}^{\infty}(4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}e^{-2k(2k+n-1)t}. \tag{1.13}$$

The Complex Projective Space $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$ The complex projective space $\mathbf{P}^n(\mathbb{C})$ is a compact rank one symmetric space in view of $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(1))$ where as usual $\mathbf{U}(n)$ denotes the group of $n \times n$ unitary matrices and $\mathbf{SU}(n)$ is the subgroup of unitary matrices with determinant one. Consideration of the long exact homotopy sequence associated with the Hopf fibration

$$\mathbb{S}^1 = \mathbf{U}(1) \hookrightarrow \mathbb{S}^{2n+1} \hookrightarrow \mathbf{P}^n(\mathbb{C}), \tag{1.14}$$

results in $\pi_1(\mathbf{P}^n(\mathbb{C})) \cong 0$ while $\pi_2(\mathbf{P}^n(\mathbb{C})) \cong \mathbb{Z}$ and $\pi_j(\mathbf{P}^n(\mathbb{C})) \cong \pi_j(\mathbb{S}^{2n+1})$ for $j \geq 3$. Thus unlike the case with the real projective space, here, $\mathscr{X}$ is simply connected. Next the spherical functions here are expressed via the Jacobi polynomials $P_k^{(\alpha,\beta)}$ $(\alpha = n-1, \beta = 0)$ as

$$\Phi_k^{\mathscr{X}}(\theta) = \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) = \frac{P_k^{(\alpha,\beta)}(\cos\theta)}{P_k^{(\alpha,\beta)}(1)}, \quad \Phi_k^{\mathscr{X}}(0) = 1, \quad P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)k!}. \tag{1.15}$$

These are in turn eigenfunctions of the Laplacian Δ, i.e., $\Delta\Phi_k^{\mathscr{X}} = k(n+k)\Phi_k^{\mathscr{X}}$ where the substitution $t = \cos\theta$ reduces to the Jacobi Eq. (1.16) [with $\alpha = n-1$, $\beta = 0$], i.e.,[2]

$$\left(1-t^2\right)\frac{d^2y}{dt^2} - (\alpha - \beta + (\alpha+\beta+2)t)\frac{dy}{dt} + k(k+\alpha+\beta+1)y = 0, \tag{1.16}$$

with the truncated power series solution $y = P_k^{(\alpha,\beta)}(t)$. Referring now to (1.3) and the above the heat kernel on the complex projective space $\mathbf{P}^n(\mathbb{C})$ admits the spectral representation

$$\begin{aligned} K_{\mathbf{P}^n(\mathbb{C})}(t,\theta) &= \frac{1}{\omega_3^n}\sum_{k=0}^{\infty} M_k^n(\mathbf{P}^n(\mathbb{C}))\Phi_k^{\mathbf{P}^n(\mathbb{C})}(\theta)e^{-k(k+n)t}, \\ &= \frac{1}{\omega_3^n}\sum_{k=0}^{\infty}\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 \mathscr{P}_k^{(n-1,0)}(\cos\theta)e^{-k(k+n)t}, \end{aligned} \tag{1.17}$$

where $\omega_3^n = \mathrm{Vol}(\mathbf{P}^n(\mathbb{C}))$ (*cf.* Table 2) and the associated heat trace is given by

$$\mathrm{tr}\, e^{-t\Delta_{\mathbf{P}^n(\mathbb{C})}} = \omega_3^n K_{\mathbf{P}^n(\mathbb{C})}(t,0) = \sum_{k=0}^{\infty}\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 e^{-k(k+n)t}, \quad t > 0. \tag{1.18}$$

The Quaternionic Projective Space $\mathscr{X} = \mathbf{P}^n(\mathbb{H})$ The quaternionic projective space provides another example of a compact rank one symmetric space in view of the identification $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$ where $\mathbf{Sp}(n)$ is the usual group of $n \times n$ symplectic matrices. The spherical functions here again are expressed via the Jacobi polynomials $P_k^{(\alpha,\beta)}$ (with $\alpha = 2n-1$, $\beta = 1$) as

[2] Note that $\mathscr{C}_k^{\nu}(t) = \mathscr{P}_k^{(\nu-1/2,\nu-1/2)}(t)$ with a similar parameter relation between the Gegenbauer and Jacobi Eqs. (1.9) and (1.16) respectively. For more on Gegenbauer and Jacobi polynomials see [3, 15, 19, 20, 33, 34].

$$\Phi_k^{\mathscr{X}}(\theta) = \mathscr{P}_k^{(2n-1,1)}(\cos\theta), \quad \Phi_k^{\mathscr{X}}(0) = 1, \quad P_k^{(2n-1,1)}(1) = \frac{\Gamma(2n+k)}{\Gamma(2n)k!}, \tag{1.19}$$

and are eigenfunctions of the Laplacian: $\Delta\Phi_k^{\mathscr{X}} = k(k+2n+1)\Phi_k^{\mathscr{X}}$. The heat kernel can be expressed through the spectral sum

$$\begin{aligned} K_{\mathbf{P}^n(\mathbb{H})}(t,\theta) &= \frac{1}{\omega_4^n}\sum_{k=0}^{\infty} M_k^n(\mathbf{P}^n(\mathbb{H}))\Phi_k^{\mathbf{P}^n(\mathbb{H})}(\theta)e^{-k(k+n)t}, \\ &= \frac{1}{\omega_4^n}\sum_{k=0}^{\infty}\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2 \mathscr{P}_k^{(2n-1,1)}(\cos\theta)e^{-k(k+2n+1)t} \end{aligned} \tag{1.20}$$

where $\omega_4^n = \mathrm{Vol}(\mathbf{P}^n(\mathbb{H}))$ (*cf.* Table 3) and the associated heat trace is given by

$$\operatorname{tr} e^{-t\Delta_{\mathbf{P}^n(\mathbb{H})}} = \sum_{k=0}^{\infty}\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{\Gamma(2n)k!}\right]^2 e^{-k(k+2n+1)t}, \qquad t>0. \tag{1.21}$$

The Cayley Projective Plane $\mathscr{X} = \mathbf{P}^2(\mathrm{Cay})$ The last example of a compact rank one symmetric space here is the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$. The spherical functions here are expressed via the Jacobi polynomials as

$$\Phi_k^{\mathscr{X}}(\theta) = \mathscr{P}_k^{(7,3)}(\cos\theta), \quad \Phi_k^{\mathscr{X}}(0) = 1, \quad P_k^{(7,3)}(1) = \frac{\Gamma(k+8)}{7!k!}, \tag{1.22}$$

and are eigenfunctions of the Laplacian: $\Delta\Phi_k^{\mathscr{X}} = k(k+11)\Phi_k^{\mathscr{X}}$. The heat kernel can be expressed through the spectral sum

$$K_{\mathbf{P}^2(\mathrm{Cay})}(t,\theta) = \frac{1}{\omega_5^n}\sum_{k=0}^{\infty} 6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}\mathscr{P}_k^{(7,3)}(\cos\theta)e^{-k(k+11)t} \tag{1.23}$$

where $\omega_5^n = \mathrm{Vol}(\mathbf{P}^2(\mathrm{Cay}))$ (*cf.* Table 3) and the associated heat trace is given by

$$\operatorname{tr} e^{-t\Delta_{\mathbf{P}^2(\mathrm{Cay})}} = \sum_{k=0}^{\infty} 6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}e^{-k(k+11)t}, \quad t>0. \tag{1.24}$$

At the end of this introduction we discuss another spectral function of huge importance, the spectral zeta function, which was first introduced and studied in [24] and has since become an indispensable tool in spectral geometry and the spectral analysis of the Laplacian. Indeed here one defines the zeta function $\zeta = \zeta_{\mathscr{X}}(s)$ associated with the compact rank one symmetric space $\mathscr{X}$ by the Dirichlet-type series

$$\zeta_{\mathscr{X}}(s) = \operatorname{tr}' \Delta_{\mathscr{X}}^{-s} = \sum_{k=1}^{\infty} M_k^n(\mathscr{X})[\lambda_k^n]^{-s}, \qquad \operatorname{Re} s > N/2, \tag{1.25}$$

where N is as illustrated in Table 1 and by the Weyl spectral asymptotic formula (see e.g., [18]) the series (1.25) converges absolutely for $\operatorname{Re} s > N/2$. As such the spectral zeta function $\zeta = \zeta_{\mathscr{X}}(s)$ is easily seen to be the Mellin transform of the heat trace $\Theta_{\mathscr{X}}(t) = \operatorname{tr} e^{-t\Delta_{\mathscr{X}}}$, that is,

$$\begin{aligned}\zeta_{\mathscr{X}}(s) &= \sum_{k=1}^{\infty} \frac{M_k^n(\mathscr{X})}{[\lambda_k^n]^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{k=1}^{\infty} M_k^n(\mathscr{X}) e^{-\lambda_k^n t} \right) dt \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\operatorname{tr} e^{-t\Delta_{\mathscr{X}}} - 1 \right) dt.\end{aligned} \tag{1.26}$$

It is well known (see, e.g., [11, 24]) that the heat trace $\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}$ admits the asymptotics

$$\begin{aligned}\Theta_{\mathscr{X}}(t) = \operatorname{tr} e^{-t\Delta_{\mathscr{X}}} &= \int_{\mathscr{X}} K_{\mathscr{X}}(t,0)\, dv_g = \int_{\mathscr{X}} K(t,x,x)\, dv_g \\ &= \sum_{k=0}^{\infty} M_k^n(\mathscr{X}) e^{-\lambda_k^n t} \sim \sum_{k=0}^{\infty} \frac{a_k^n t^k}{(4\pi t)^{N/2}}, \quad t \searrow 0,\end{aligned} \tag{1.27}$$

where $a_k^n = a_k^n(\mathscr{X})$ are called the Minakshisundaram–Pleijel coefficients or simply the heat coefficients associated with the n-space $\mathscr{X}$. It now follows that the zeta function $\zeta_{\mathscr{X}}(s)$ can be continued to a meromorphic function to all of $\mathbb{C}$ with its poles all being simple and located on the real axis at

$$s_k = N/2 - k, k = 0, 1, 2, \ldots \qquad N \text{ odd}, \tag{1.28}$$

$$s_k = N/2 - k, k = 0, 1, 2, \ldots, N/2 - 1 \quad N \text{ even}, \tag{1.29}$$

(here N is the dimension of $\mathscr{X}$ as illustrated in Table 1). Furthermore here the residues are given explicitly by the formula

$$\operatorname{Res} \zeta_{\mathscr{X}}(s) \Big|_{s=s_k} = \frac{a_k^n}{(4\pi)^{\frac{N}{2}} \Gamma\left(\frac{N}{2} - k\right)}, \tag{1.30}$$

which shows the intimate relation between the heat coefficients and the spectral residues; we discuss these more later on. (For further reading and related discussions including motivation and various applications see [2, 3, 13, 18] as well as [6, 8, 9, 12, 24, 25, 27, 30, 31–33].)

2 The Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t, \theta)$

In this section we examine the Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t, \theta)$ with respect to the θ-variable near the origin $\theta = 0$ (for $t > 0$). A more refined analysis and description of the resulting Maclaurin heat coefficients including relationships to other heat invariants then follow in the subsequent sections. It is interesting to note that the Maclaurin expansion of $K_{\mathscr{X}}(t,\theta)$ can be purely expressed in terms of the heat trace and its higher order derivatives and that the heat trace alone determines the Maclaurin heat coefficients. Towards this end recall that the Maclaurin expansion of $K_{\mathscr{X}}(t,\theta)$ about $\theta=0$ has the form

$$\begin{aligned} K_{\mathscr{X}}(t,\theta) &= \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \left\{ \frac{\partial^{2j}}{\partial\theta^{2j}} K_{\mathscr{X}}(t,\theta)\bigg|_{\theta=0} \right\} = K_{\mathscr{X}}(t,0) + \frac{\theta^2}{2!}\frac{\partial^2}{\partial\theta^2} K_{\mathscr{X}}(t,0) \\ &+ \frac{\theta^4}{4!}\frac{\partial^4}{\partial\theta^4} K_{\mathscr{X}}(t,0) + \cdots + \frac{\theta^{2m}}{(2m)!}\frac{\partial^{2m}}{\partial\theta^{2m}} K_{\mathscr{X}}(t,0) + \cdots . \end{aligned} \tag{2.1}$$

Note that in view of $K_{\mathscr{X}}(t,\theta)$ being even in the θ-variable [*cf.* (1.3)] all partial derivatives of odd order vanish at $\theta=0$ and hence the Maclaurin expansion contains only even terms. Evidently the first term in (2.1) is given by the usual trace formula, namely

$$K_{\mathscr{X}}(t,0) = \frac{1}{\mathrm{Vol}(\mathscr{X})} \sum_{k=0}^{\infty} M_k^n(\mathscr{X}) e^{-\lambda_k^n t} = \frac{\mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}}}{\mathrm{Vol}(\mathscr{X})}. \tag{2.2}$$

The following proposition whose proof is a consequence of the Leibniz rule and an induction argument (see [3] for a proof and Appendix C for the computation of the first few coefficients) will play an important role later on. The first few Jacobi coefficients $c_j^\ell(\alpha,\beta)$ are described in Table 4 (see also the Appendix for the supporting calculations).

Proposition 2.1 (Jacobi Coefficients) *Let* $\mathscr{P}_k^{(\alpha,\beta)}(t) = \Phi_k^{\mathscr{X}}(\arccos t)$ *be spherical functions on* $\mathscr{X}$. *Then*

$$\frac{d^{2\ell}}{d\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\bigg|_{\theta=0} = \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta)[k(k+\alpha+\beta+1)]^j, \quad \ell \geq 1; \alpha,\beta > -1. \tag{2.3}$$

The scalars $c_j^\ell(\alpha,\beta)$ *are called the Jacobi coefficients while* $\lambda_k^n := \lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1)$ *with* $k \geq 0$ *are the eigenvalues of the Jacobi operator.*

We now apply Proposition 2.1 to compute even derivatives of the heat kernel $K_{\mathscr{X}}(t,\theta)$ and subsequently determine the Maclaurin expansion given by (2.1) as a

Table 4 The Jacobi coefficients $c_j^\ell(\alpha,\beta)$ associated with the symmetric space $\mathscr{X}$

j	$\ell=1$	2	3
1	$-\frac{1}{2(\alpha+1)}$	$-\frac{(\alpha+3\beta+2)}{4(\alpha+1)(\alpha+2)}$	$-\frac{4(\alpha^2+5\alpha+6)+30\beta(\beta+\alpha+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)}$
2		$\frac{3}{4(\alpha+1)(\alpha+2)}$	$\frac{15\alpha+45\beta+30}{8(\alpha+1)(\alpha+2)(\alpha+3)}$
3			$-\frac{15}{8(\alpha+1)(\alpha+2)(\alpha+3)}$

spectral series involving the trace of the associated heat operator and its higher derivatives. Indeed by (1.3)

$$\frac{\partial^{2\ell}}{\partial\theta^{2\ell}} K_{\mathscr{X}}(t,\theta)\bigg|_{\theta=0} = \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} e^{-\lambda_k^n t} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} \Phi_k^{\mathscr{X}}(\theta)\bigg|_{\theta=0}, \tag{2.4}$$

and as a consequence of Proposition (2.1) we have

$$\begin{aligned}(2.4) &= \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} e^{-\lambda_k^{\alpha,\beta} t} \frac{\partial^{2\ell}}{\partial\theta^{2\ell}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\bigg|_{\theta=0} \\ &= \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})} e^{-\lambda_k^{\alpha,\beta} t} \sum_{j=1}^{\ell} c_j^{\ell}(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^j = \sum_{j=1}^{\ell} \frac{c_j^{\ell}(\alpha,\beta)}{\mathrm{Vol}(\mathscr{X})}\left(-\frac{d}{dt}\right)^j \mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}}.\end{aligned} \tag{2.5}$$

This nicely reveals that the Maclaurin expansion of $K_{\mathscr{X}}(t,\theta)$ can be obtained solely from the heat trace and its higher derivatives and that the diagonal part of the heat kernel, that is, $K_{\mathscr{X}}(t,0) = \mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}}/\mathrm{Vol}(\mathscr{X})$ determines its off-diagonal part $K_{\mathscr{X}}(t,\theta)$ $(\theta \sim 0)$.

Theorem 2.2 (Spectral expansion) *The Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t,\theta)$ associated with the compact rank one symmetric space $\mathscr{X}$ given by (2.1) admits the formulation*

$$K_{\mathscr{X}}(t,\theta) = \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!}\left\{\frac{\partial^{2m}}{\partial\theta^{2m}} K_{\mathscr{X}}(t,\theta)\bigg|_{\theta=0}\right\} = \sum_{\ell=0}^{\infty} \frac{b_{2\ell}^n(t)}{\mathrm{Vol}(\mathscr{X})}\theta^{2\ell}, \tag{2.6}$$

where $b_{2\ell}^n = b_{2\ell}^n(t)$ $(t>0, \ell \geq 0)$ the Maclaurin heat coefficients associated with $\mathscr{X}$, can be expressed as

$$b_0^n(t) = \mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}}, \quad b_{2\ell}^n(t) = \mathscr{R}_\ell\left(-\frac{d}{dt}\right)\mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}} \quad \ell \geq 1. \tag{2.7}$$

Here $\mathscr{R}_\ell = \mathscr{R}_\ell(\mathrm{X})$ $(\text{with} \ell \geq 1)$ is the ℓth degree polynomial built out of the Jacobi coefficients $(c_j^{\ell}(\alpha,\beta); 1 \leq j \leq \ell)$ as in Proposition 2.1 given by

$$\mathscr{R}_\ell(\mathrm{X}) = \sum_{j=1}^{\ell} \frac{c_j^{\ell}(\alpha,\beta)}{(2\ell)!}\mathrm{X}^j. \tag{2.8}$$

Thus referring to Table 4 the Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t,\theta)$ admits the following formulation in terms of the trace of the heat operator:

$$
\begin{aligned}
K_{\mathscr{X}}(t,\theta) =& \Bigg\{ 1 + \frac{\theta^2/2!}{2(\alpha+1)}\frac{d}{dt} + \frac{\theta^4/4!}{4(\alpha+1)(\alpha+2)}\left[(\alpha+3\beta+2)\frac{d}{dt} + 3\frac{d^2}{dt^2}\right] \\
&+ \frac{\theta^6/6!}{8(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[\left(4\left(\alpha^2+5\alpha+6\right)+30\beta(\beta+\alpha+2)\right)\frac{d}{dt} \\
&+ (15\alpha+45\beta+30)\frac{d^2}{dt^2} + 15\frac{d^3}{dt^3}\Bigg] + \cdots \Bigg\} \frac{\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}}{\operatorname{Vol}(\mathscr{X})}.
\end{aligned} \tag{2.9}
$$

In particular we have $b_0^n(t) = \operatorname{tr} e^{-t\Delta_{\mathscr{X}}}$ and for $\ell \geq 1$

$$
b_2^n(t) = \frac{1/2!}{2(\alpha+1)}\frac{d}{dt}\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}, \tag{2.10}
$$

$$
b_4^n(t) = \frac{1/4!}{4(\alpha+1)(\alpha+2)}\left[(\alpha+3\beta+2)\frac{d}{dt} + 3\frac{d^2}{dt^2}\right]\operatorname{tr} e^{-t\Delta_{\mathscr{X}}} \tag{2.11}
$$

$$
\begin{aligned}
b_6^n(t) =& \frac{1/6!}{8(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg[\left(4\left(\alpha^2+5\alpha+6\right)+30\beta(\beta+\alpha+2)\right)\frac{d}{dt} \\
&+ (15\alpha+45\beta+30)\frac{d^2}{dt^2} + 15\frac{d^3}{dt^3}\Bigg]\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}
\end{aligned} \tag{2.12}
$$

and so on. (For the supporting calculation relating to Table 4 see the Appendix at the end.) As the spherical functions on the real projective space $\mathbf{P}^n(\mathbb{R})$ can be obtained from those on the sphere $\mathbb{S}^n$ by changing k to $2k$ in the Gegenbauer differential Eq. (1.9) we only give explicit examples of the heat invariants and heat coefficients associated with $\mathbb{S}^n$ and $\mathbf{P}^n(\mathbb{C})$. Note that when the underlying symmetric space is the sphere $\mathscr{X} = \mathbb{S}^n$ or the real projective space $\mathscr{X} = \mathbf{P}^n(\mathbb{R})$ the Jacobi coefficients $c_j^\ell(\alpha,\beta)$ correspond to $\alpha = \beta = \nu - 1/2 = (n-2)/2$; in particular the two share the same coefficients as the Jacobi coefficient is independent of k. When $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$ the Jacobi coefficients $c_j^\ell(\alpha,\beta)$ correspond to $\alpha = n-1,\ \beta = 0$.

Specialising now to the sphere $\mathscr{X} = \mathbb{S}^n$ with $c_j^\ell(\alpha,\beta) = c_j^\ell((n-2)/2,(n-2)/2)$ we have

$$
\begin{aligned}
K_{\mathbb{S}^n}(t,\theta) =& \Bigg\{ 1 + \frac{\theta^2}{2!}\frac{1}{n}\frac{d}{dt} + \frac{\theta^4}{4!}\frac{1}{n(n+2)}\left(2(n-1)\frac{d}{dt} + 3\frac{d^2}{dt^2}\right) + \frac{\theta^6}{6!}\frac{1}{n(n+2)(n+4)} \\
&\times \left(8(2n-1)(n-1)\frac{d}{dt} + 30(n-1)\frac{d^2}{dt^2} + 15\frac{d^3}{dt^3}\right) + \cdots \Bigg\} \frac{\operatorname{tr} e^{-t\Delta_{\mathbb{S}^n}}}{\omega_1^n},
\end{aligned} \tag{2.13}
$$

while for the complex projective space $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$ with $c_j^\ell(\alpha,\beta) = c_j^\ell(n-1,0)$ we have

$$K_{\mathbf{P}^n(\mathbb{C})}(t,\theta) = \left\{1 + \frac{\theta^2}{2!}\frac{1}{2n}\frac{d}{dt} + \frac{\theta^4}{4!}\frac{1}{4n}\left(\frac{d}{dt} + \frac{3}{n+1}\frac{d^2}{dt^2}\right) + \frac{\theta^6}{6!}\frac{1}{8n}\right.$$
$$\left. \times \left(4\frac{d}{dt} + \frac{15}{(n+2)}\frac{d^2}{dt^2} + \frac{15}{(n+1)(n+2)}\frac{d^3}{dt^3}\right) + \cdots \right\} \frac{\operatorname{tr} e^{-t\Delta_{\mathbf{P}^n(\mathbb{C})}}}{\omega_3^n}. \tag{2.14}$$

3 Asymptotics of the Maclaurin spectral functions $b^n_{2\ell}(t)$ and the generalised heat coefficients

In this section we describe the asymptotics ($t \searrow 0$) of the Maclaurin spectral functions $b^n_{2\ell}(t)$ associated with the heat kernel $K_{\mathscr{X}}(t,\theta)$ ($t > 0$) [*cf.* (2.1)] in terms of the Minakshisundaram–Pleijel heat coefficients $a^n_k = a^n_k(\mathscr{X})$ which on the one hand connects to the spectral zeta-function of $\mathscr{X}$ and its residues and on the other hand leads to a new set of heat coefficients associated with the symmetric space $\mathscr{X}$. (See [2, 3] and for more on heat coefficients and applications see [6, 14, 21, 23, 24, 28, 29] as well as [1, 9, 13, 25, 27, 30, 31, 32].)

Here the generalised Minakshisundaram–Pleijel heat coefficients $(a^{n,\ell}_{k,j}(\mathscr{X}) : k \geq 0, 1 \leq j \leq \ell)$ arise from the heat trace asymptotics (1.27). Indeed for any integer $j \geq 1$ upon taking the jth derivative of the heat trace as expressed above we have

$$\left(\frac{d}{dt}\right)^j \sum_{k=0}^{\infty} \frac{a^n_k t^k}{(4\pi t)^{\frac{N}{2}}} = \sum_{k=0}^{\infty} \frac{a^n_k}{(4\pi)^{\frac{N}{2}}}\left(\frac{d}{dt}\right)^j t^{k-\frac{N}{2}} = \sum_{k=0}^{\infty} \frac{\left(k-\frac{N}{2}\right)! a^n_k t^{k-\frac{N}{2}-j}}{(4\pi)^{\frac{N}{2}}\left(k-\frac{N}{2}-j\right)!}. \tag{3.1}$$

(Note that for non-integers factorial is defined via the usual Gamma function.) Combining the above with the description of the Macluarin heat coefficients from Theorem 2.2 leads to the following statement.

Theorem 3.1 *Consider the Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t,\theta)$ ($t > 0$) as given by (2.1). Then the spectral functions $b^n_{2\ell} = b^n_{2\ell}(t)$ in (2.7) admit the asymptotics*

$$b^n_{2\ell}(t) = \sum_{j=1}^{\ell} \frac{c^\ell_j(\alpha,\beta)}{(2\ell)!}\left(-\frac{d}{dt}\right)^j \operatorname{tr} e^{-t\Delta_{\mathscr{X}}} \sim \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{k=0}^{\infty}\sum_{j=1}^{\ell} a^{n,\ell}_{k,j} t^{k-j}, \quad \ell \geq 1, \tag{3.2}$$

as $t \searrow 0$ where we have

$$a^{n,\ell}_{k,j} = (-1)^j \frac{c^\ell_j(\alpha,\beta)\left(k-\frac{N}{2}\right)!}{(2\ell)!\left(k-\frac{N}{2}-j\right)!} a^n_k, \quad a^{n,0}_{k,0} = a^n_k. \tag{3.3}$$

In analogy with (1.27) the coefficients $(a^{n,\ell}_{k,j})$ are hereafter referred to as the generalised Minakshisundaram–Pleijel heat coefficients. We next give a more explicit description of the asymptotics of the spectral function $b_{2\ell} = b_{2\ell}(t)$ in terms of the Minakshisundaram–Pleijel heat coefficients and the spectral residues associated with the symmetric space $\mathscr{X}$. This serves as an illustration of the above theorem and we return to a detailed analysis of these functions and their asymptotics later in Sect. 4. For the sake of brevity and definiteness the following is confined to $\mathscr{X} = \mathbb{S}^n$ and $\mathbf{P}^n(\mathbb{C})$ only but the other cases are similar.

The Generalised Heat Coefficients $a_{k,j}^{n,\ell}(\mathscr{X})$ Here we express the asymptotics of the spectral functions $b_{2\ell}^n = b_{2\ell}^n(t)$ by using the explicit form of the heat coefficients $a_k^n = a_k^n(\mathscr{X})$. The formulation of the results below is confined to $\mathscr{X} = \mathbb{S}^n$ but the other cases are similar.

(a) $\mathscr{X} = \mathbb{S}^n : n \geq 3$ Odd. For the odd dimensional sphere $\mathbb{S}^n$ the Minakshisundaram–Pleijel coefficients can be calculated using a variety of techniques and are given by (see, e.g., [3])

$$a_k^n = \sum_{m=1}^{\frac{n-1}{2}} (4\pi)^{n/2} \frac{[(n-1)/2]^{2k+2m-n+1}\Gamma(m+1/2)}{(k-(n-1)/2+m)!(n-1)!} \mathsf{E}_m^n; \tag{3.4}$$

where the scalars E_m^n are the coefficients of the polynomial $\mathscr{A} = \mathscr{A}(X)$ as defined below

$$\mathscr{A}(X) = \prod_{j=0}^{s} \left(X^2 - j^2\right) = \sum_{m=1}^{s-1} \mathsf{E}_m^n X^{2m}, \quad s = (n-3)/2. \tag{3.5}$$

Now referring to (3.3) (with $N = n$) and (3.4) the generalised Minakshisundaram–Pleijel coefficients can be expressed as

$$a_{k,j}^{n,\ell} = (-1)^\ell (4\pi)^{\frac{n}{2}} \frac{c_j^\ell(\alpha,\beta)\left(k-\frac{n}{2}\right)!}{(2\ell)!\left(k-\frac{n}{2}-j\right)!} \sum_{m=1}^{\frac{n-1}{2}} \frac{\left(\frac{n-1}{2}\right)^{2k+2m-n+1}\Gamma(m+1/2)}{\left(k-\frac{n-1}{2}+m\right)!(n-1)!} \mathsf{E}_m^n. \tag{3.6}$$

For the sake of clarification and further illustration let us pause briefly to examine some of the low dimensional cases. Indeed from the formulation (3.4) (with $n = 3, 5$ and 7) it is easily seen that (for $k \geq 0$) we have

$$\begin{aligned} a_k^3 &= \frac{2\pi^2}{k!}, \quad a_k^5 = \frac{2^{2k-1}\pi^3}{3\cdot k!}(6-k), \\ a_k^7 &= \frac{3^{2k-6}\pi^4}{5\cdot k!}\left(16k^2 - 286k + 1215\right). \end{aligned} \tag{3.7}$$

In case of $\mathbb{S}^3$ using (3.3) and the first identity in (3.7) the generalised Minakshisundaram–Pleijel coefficients can be seen to be

$$a_{k,j}^{3,\ell} = 2\pi^2 \frac{c_j^\ell(1/2,1/2)}{(2\ell)!} \frac{(-1)^\ell (k-3/2)!}{k!(k-3/2-j)!}. \tag{3.8}$$

Hence upon invoking (3.2) the asymptotics of the spectral functions $b_{2\ell}^3(t)$ (as $t \searrow 0$) can be obtained as ($\ell \geq 1$)

$$b^3_{2\ell}(t) = \mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta_{\mathbb{S}^3}} \sim \sum_{j=1}^{\ell}\sum_{k=0}^{\infty}(-1)^j \frac{c_j^\ell(1/2,1/2)\sqrt{\pi}}{4\cdot(2\ell)!}\frac{\left(k-\frac{3}{2}\right)!}{\left(k-\frac{3}{2}-j\right)!}t^{k-3/2-j}. \tag{3.9}$$

This together with the explicit Jacobi coefficients from Table 4 [compare with (2.10) and (4.7)–(4.9)] gives the first few spectral functions asymptotics as

$$\begin{aligned}
b^3_2(t) &\sim \frac{4\pi^3/3}{(4\pi t)^{\frac{5}{2}}}\sum_{k=0}^{\infty}\frac{k-3/2}{k!}t^k,\\
b^3_4(t) &\sim \frac{4\pi^4/15}{(4\pi t)^{\frac{7}{2}}}\sum_{k=0}^{\infty}\left(\frac{4}{3}\frac{k-3/2}{k!}t+\frac{k^2-4k+15/4}{k!}\right)t^k,\\
b^3_6(t) &\sim \frac{8\pi^5/315}{(4\pi t)^{\frac{9}{2}}}\sum_{k=0}^{\infty}\left(\frac{16}{3}\frac{k-3/2}{k!}t^2+4\frac{k^2-4k+15/4}{k!}t+\right.\\
&\qquad\left.+\frac{k^3-15k^2/2+71k/4-105/8}{k!}\right)t^k.
\end{aligned}$$

Theorem 3.2 *The spectral functions $b^n_{2\ell}=b^n_{2\ell}(t)$ associated with the odd dimensional sphere $\mathbb{S}^n$ admit the asymptotics ($t\searrow 0$)*

$$b^n_{2\ell}(t) \sim \sum_{k=0}^{\infty}\sum_{j=1}^{\ell}(-1)^j\frac{c_j^\ell(\alpha,\beta)\left(k-\frac{n}{2}\right)!}{(2\ell)!\left(k-\frac{n}{2}-j\right)!}\sum_{m=1}^{\frac{n-1}{2}}\frac{\left(\frac{n-1}{2}\right)^{2k+2m-n+1}\Gamma\left(m+\frac{1}{2}\right)}{\left(k-\frac{n-1}{2}+m\right)!(n-1)!}\mathsf{E}^n_m t^{k-\frac{n}{2}-j}. \tag{3.10}$$

Here $\alpha=\beta=(n-2)/2$ and the scalars E^n_m are as in (3.5).

Referring to the discussion prior to the theorem and the formulation (3.7) we have the counterpart for $n=5$ and 7 with $\ell\geq 1$ as below:

$$\begin{aligned}
b^5_{2\ell}(t) &\sim \sum_{k=0}^{\infty}\sum_{j=1}^{\ell}(-1)^j\frac{c_j^\ell(3/2,3/2)2^{2j-1}\sqrt{\pi}}{3\cdot 32\cdot(2\ell)!}2^{2k}\frac{\left(k-\frac{5}{2}\right)!(6-k)}{k!\left(k-\frac{5}{2}-j\right)!}t^{k-5/2-j}.\\
b^7_{2\ell}(t) &\sim \sum_{k=0}^{\infty}\sum_{j=1}^{\ell}(-1)^j\frac{c_j^\ell(5/2,5/2)\sqrt{\pi}}{5\cdot 128\cdot(2\ell)!}3^{2k-6}\frac{(16k^2-286k+1215)(k-7/2)!}{k!(k-7/2-j)!}t^{k-\frac{7}{2}-j}.
\end{aligned} \tag{3.11}$$

(b) $\mathscr{X}=\mathbb{S}^n : n\geq 2$ Even. For the even dimensional sphere $\mathbb{S}^n$ the Minakshisundaram–Pleijel coefficients can again be calculated using a variety of methods and are given by (see, e.g., [3])

$$a_k^n = \frac{(4\pi)^{n/2}}{\Gamma(n)} \left\{ \sum_{m=0}^{\frac{n-2}{2}} \frac{(\frac{n}{2}-m-1)!}{(k-m)!} \left(\frac{n-1}{2}\right)^{2k-2m} \mathsf{F}_m^n \right.$$
$$\left. + \sum_{m=0}^{\frac{n-2}{2}} \mathsf{F}_m^n \sum_{q=\frac{n}{2}-m}^{k-m} \frac{(-1)^{q+\frac{n}{2}-m-1}(2^{1-2q}-1)}{q(k-m-q)!(q-\frac{n}{2}+m)!} \left(\frac{n-1}{2}\right)^{2k-2m-2q} B_{2q} \right\}. \tag{3.12}$$

Note that in (3.12) B_{2l} are the well-known Bernoulli numbers (*see* [15]) and the scalars F_m^n are the coefficients of the polynomial $\mathscr{B} = \mathscr{B}(X)$ as defined below

$$\mathscr{B}(X) = \prod_{j=\frac{1}{2}}^{s-\frac{1}{2}} \left(X^2 - j^2\right) = \sum_{m=0}^{s} \mathsf{F}_m^n X^{n-2m-2}, \quad s = (n-2)/2. \tag{3.13}$$

Theorem 3.3 $(n \geq 2\,\text{even})$ *The spectral functions $b_{2\ell}^n = b_{2\ell}^n(t)$ associated with the even dimensional sphere $\mathbb{S}^n$ admit the asymptotics $(t \searrow 0)$*

$$b_{2\ell}^n(t) \sim \frac{1}{(4\pi)^{n/2}} \sum_{k=0}^{\infty} \sum_{j=1}^{\ell} (-1)^j \frac{c_j^\ell(\alpha,\beta)\left(k-\frac{n}{2}\right)!}{(2\ell)!\left(k-\frac{n}{2}-j\right)!} a_k^n t^{k-\frac{n}{2}-j}. \tag{3.14}$$

Here a_k^n are as in (3.12) and again $\alpha = \beta = (n-2)/2$.

The Spectral Residues $\mathrm{Res}\,\zeta_{\mathscr{X}}$ The asymptotics of the spectral function $b_{2\ell}^n(t)$ can also be expressed in terms of the residues of the spectral zeta function $\zeta = \zeta_{\mathscr{X}}$ [see (1.25)] via the relation (1.30). Rather than giving the explicit asymptotics here we present the spectral residues for $\zeta_{\mathscr{X}}$ in the cases of spheres and projective spaces. (See [2, 3] for further detail and more.)

- $\mathscr{X} = \mathbb{S}^n$: Regardless of whether n is even or odd the multiplicity function $M_k^n(\mathbb{S}^n)$ can be written for suitable choice of coefficients G_m^n as

$$M_k^n = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!} = \frac{1}{(n-1)!} \sum_{m=0}^{n-1} \mathsf{G}_m^n \left(k + \frac{n-1}{2}\right)^m. \tag{3.15}$$

  Making use of (1.25) and (1.30) and setting $L = \min\left(k, \left[\frac{n-2}{2}\right]\right)$, where $[x]$ is the usual *integer* part of x, it can be seen that [3]

$$\mathrm{Res}\,\zeta_{\mathbb{S}^n}(s)\Big|_{s=n/2-k} = \frac{1}{2} \sum_{\ell=0}^{L} (-1)^{k-\ell} \frac{\mathsf{G}_{n-2\ell-1}^n}{(k-1)!} \left(\frac{n-1}{2}\right)^{2k-2\ell} \binom{k-n/2}{k-\ell}, \quad k \geq 0. \tag{3.16}$$

- $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$: The multiplicity function $M_k^n(\mathbf{P}^n(\mathbb{C}))$ (*cf.* Table 2) can be written in a polynomial form for suitable choice of coefficients H_m^n and according to whether n is odd or even as:

$$\begin{aligned} M_k^n &= \frac{2(k+n/2)}{n!(n-1)!} \prod_{j=0}^{(n-3)/2} \left[(k+n/2)^2 - (j+1/2)^2\right]^2 \quad n \geq 3 \text{ odd} \\ &= \frac{2(k+n/2)^3}{n!(n-1)!} \prod_{j=1}^{(n-2)/2} \left[(k+n/2)^2 - j^2\right]^2 \quad n \geq 4 \text{ even} \\ &= (k+n/2) \sum_{m=0}^{n-1} \mathsf{H}_m^n [k(k+n)]^m. \end{aligned} \tag{3.17}$$

Inserting the multiplicity (3.17) into the spectral zeta function (1.25) and using (1.30) it follows that (see [2])

$$\operatorname{Res} \zeta_{\mathbf{CP}^n}(s)\Big|_{s=n-k} = \frac{1}{2}\mathsf{H}_{n-k-1}^n, \quad k = 0, 1, 2, \ldots n-1. \tag{3.18}$$

4 A Jacobi function description of the spectral functions $b_{2\ell}^n(t)$ and their asymptotics as $t \searrow 0$

This final section of the paper uses the explicit formulation of the heat kernel $K_{\mathscr{X}}(t,\theta)$ and the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n(t)$ in terms of the Jacobi theta functions $\vartheta_1, \vartheta_2, \vartheta_3$ to give a power series *vs* polynomial description of the spectral functions $b_{2\ell}^n(t)$ and the heat kernel $K_{\mathscr{X}}(t,\theta)$ as $t \searrow 0$ and $\theta \sim 0$. Towards this end recall that the Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t,\theta)$ can be written in terms of the heat trace (*cf.* Theorem 2.2) as

$$\begin{aligned} K_{\mathscr{X}}(t,\theta) &= \sum_{\ell=0}^{\infty} \frac{b_{2\ell}^n(t)}{\operatorname{Vol}(\mathscr{X})}\theta^{2\ell} = \frac{\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}}{\operatorname{Vol}(\mathscr{X})} + \sum_{\ell=1}^{\infty} \mathscr{R}_\ell\left(-\frac{d}{dt}\right)\frac{\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}}{\operatorname{Vol}(\mathscr{X})}\theta^{2\ell} \\ &= \mathbf{I} + \theta^2 \mathbf{II} + \theta^4 \mathbf{III} + \cdots, \end{aligned} \tag{4.1}$$

where the polynomials $\mathscr{R}_\ell = \mathscr{R}_\ell(X)$ are as in (2.8) while here we have set

$$\begin{aligned} \mathbf{I} &= \frac{b_0^n(t)}{\operatorname{Vol}(\mathscr{X})} = \frac{\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}}{\operatorname{Vol}(\mathscr{X})}, \quad \mathbf{II} = \frac{b_2^n(t)}{\operatorname{Vol}(\mathscr{X})} = \frac{1}{2!}\frac{1/2(\alpha+1)}{\operatorname{Vol}(\mathscr{X})}\frac{d}{dt}\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}, \\ \mathbf{III} &= \frac{b_4^n(t)}{\operatorname{Vol}(\mathscr{X})} = \frac{1}{4!}\frac{1/2(\alpha+2)}{2(\alpha+1)\operatorname{Vol}(\mathscr{X})}\left((\alpha+3\beta+2)\frac{d}{dt} + 3\frac{d^2}{dt^2}\right)\operatorname{tr} e^{-t\Delta_{\mathscr{X}}}. \end{aligned} \tag{4.2}$$

For the sake of definiteness we now specialise the discussion to some of the symmetric spaces in the list and investigate further the asymptotics of the spectral functions $b_{2\ell}^n$ by expressing the heat kernel via Jacobi theta functions. This then leads to a novel set of polynomials *vs* power series that encode spectral and asymptotic features of the heat kernel.

The Sphere $\mathscr{X} = \mathbb{S}^n$ Here we give an asymptotic description of the heat kernel $K_{\mathbb{S}^n}(t, \theta)$ $(t > 0)$ for when $t \searrow 0$ and $\theta \sim 0$ via the Jacobi theta functions ϑ_1, ϑ_2 and a further asymptotic analysis is presented which results in a set of heat related polynomials and power series (in the case of n odd *vs* even). As the formulation of the heat kernel in terms of the Jacobi functions varies depending on whether n is odd or even we proceed accordingly. Recall that the heat kernel $K_{\mathbb{S}^n}(t, \theta)$ admits the Maclaurin expansion

$$\begin{aligned} K_{\mathbb{S}^n}(t, \theta) &= \sum_{\ell=0}^{\infty} \frac{b_{2\ell}^n(t)}{\omega_1^n} \theta^{2\ell} = \frac{1}{\omega_1^n} \operatorname{tr} e^{-t\Delta} + \frac{1}{\omega_1^n} \sum_{\ell=1}^{\infty} \mathscr{R}_\ell \left(-\frac{d}{dt} \right) \operatorname{tr} e^{-t\Delta} \theta^{2\ell} \\ &= \mathbf{I} + \theta^2 \mathbf{II} + \theta^4 \mathbf{III} + \cdots, \end{aligned} \tag{4.3}$$

where using the associated Jacobi coefficients $c_j^\ell(\alpha, \beta)$ (with $\alpha = \beta = (n-2)/2$) we have

$$\mathbf{I} = \frac{1}{\omega_1^n} \operatorname{tr} e^{-t\Delta}, \quad \mathbf{II} = \frac{1}{2! n \omega_1^n} \frac{d}{dt} \operatorname{tr} e^{-t\Delta},$$
$$\mathbf{III} = \frac{1/(n+2)}{4! n \omega_1^n} \left(2(n-1) \frac{d}{dt} + 3 \frac{d^2}{dt^2} \right) \operatorname{tr} e^{-t\Delta}.$$

(a) Odd $n \geq 3$. For the sake of future calculations we proceed by first writing the multiplicity $M_k^n(\mathbb{S}^n)$ in a polynomial form (see the Appendix for notation and detail)

$$\begin{aligned} M_k^n &= \frac{2k + n - 1}{(n-1)!} \prod_{j=1}^{n-2} (k + j) = \frac{2}{(n-1)!} \prod_{j=0}^{s} (X_k^2 - j^2) \\ &= \frac{2}{(n-1)!} \sum_{m=0}^{s} \mathsf{A}_m^n X_k^{2m+2} = \frac{2\mathscr{A}(X_k)}{(n-1)!} \qquad s = (n-3)/2, \end{aligned} \tag{4.4}$$

where $\mathscr{A} = \mathscr{A}_n$ is the degree $n-1$ polynomial as in (B.1) and $X_k = k + (n-1)/2$ with $k \geq 0$. Substituting for $M_k^n(\mathbb{S}^n)$ in the heat trace formula (1.10) and using the root structure of $\mathscr{A}(X)$ it follows that the latter can be expressed as

$$\begin{aligned} \Theta_{\mathbb{S}^n}(t) = \operatorname{tr} e^{-t\Delta_{\mathbb{S}^n}} &= 2 \frac{e^{\frac{(n-1)^2}{4} t}}{(n-1)!} \sum_{k=0}^{\infty} \mathscr{A}(X_k) e^{-X_k^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4} t}}{(n-1)!} \sum_{m=0}^{s} \mathsf{A}_m^n \sum_{X_k=1}^{\infty} 2 X_k^{2m+2} e^{-X_k^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4} t}}{(n-1)!} \sum_{m=0}^{s} (-1)^{m+1} \mathsf{A}_m^n \vartheta_1^{(m+1)}(t). \end{aligned} \tag{4.5}$$

Here $\vartheta_1^{(m+1)}$ (with $m \geq 0$) denote the respective derivatives of the Jacobi theta function ϑ_1 (see the Appendix at the end for notation). Substituting (4.5) into (2.7) in Theorem 2.2 and setting $\alpha = \beta = (n-2)/2$ in the Jacobi coefficients it follows that

$$b_{2\ell}^n(t) = \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!} \sum_{j=1}^{\ell} \sum_{r=0}^{j} \frac{c_j^\ell(\alpha,\beta)}{(2\ell)!} \binom{j}{r} \left(\frac{n-1}{2}\right)^{2r} \sum_{m=0}^{s} (-1)^{m+j+1} \mathsf{A}_m^n \vartheta_1^{(m+j-r+1)}(t). \tag{4.6}$$

This above formulation upon invoking the asymptotics of the Jacobi theta function ϑ_1 as given by (A.1) and (A.2) leads to the following conclusion.

Theorem 4.1 (Odd $n \geq 3$) *The spectral functions $b_{2\ell}^n = b_{2\ell}^n(t)$ associated with $\mathscr{X} = \mathbb{S}^n$ for $n \geq 3$ odd admit the asymptotics as $t \searrow 0$*

$$b_{2\ell}^n(t) = \mathscr{R}_\ell\left(-\frac{d}{dt}\right) \operatorname{tr} e^{-t\Delta} \sim \omega_1^n \frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}+\ell}} P_{2\ell}^n(t), \tag{4.7}$$

where $P_{2\ell} = P_{2\ell}(t)$ with $\ell \geq 0$ are the polynomials

$$P_0^n(t) = 1 + \frac{1}{(n-2)!!} \sum_{m=0}^{s-1} \mathsf{A}_m^n (2m+1)!!(2t)^{s-m} \tag{4.8}$$

$$\begin{aligned} P_{2\ell}^n(t) = & \frac{2^\ell \pi^\ell}{(n-2)!!} \sum_{j=1}^{\ell} \sum_{r=0}^{j} \frac{c_j^\ell(\alpha,\beta)}{(2\ell)!} \binom{j}{r} \left(\frac{n-1}{2}\right)^{2r} \\ & \times \sum_{m=0}^{s} (-1)^r \mathsf{A}_m^n (2m+2j-2r+1)!!(2t)^{\ell+s-m-j+r}, \quad \ell \geq 1. \end{aligned} \tag{4.9}$$

Here s is as in (4.4), A_m^n are as in (B.1) and $c_j^\ell = c_j^\ell(\alpha,\beta)$ are the Jacobi coefficients with $\alpha = \beta = (n-2)/2$.

Inserting the asymptotics (4.7) with (4.8) and (4.9) into (4.3) we obtain the following formulation of the asymptotics of $K_{\mathbb{S}^n}(t,\theta)$ in terms of the polynomials $P_{2\ell}^n(t)$:

$$\begin{aligned} K_{\mathbb{S}^n}(t,\theta) &= \frac{1}{\omega_1^n} \operatorname{tr} e^{-t\Delta} + \frac{1}{\omega_1^n} \sum_{\ell=1}^{\infty} \mathscr{R}_\ell\left(-\frac{d}{dt}\right) \operatorname{tr} e^{-t\Delta} \theta^{2\ell} \\ &\sim \frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}} \left\{ P_0^n(t) + \frac{P_2^n(t)}{(4\pi t)} \theta^2 + \frac{P_2^n(t)}{(4\pi t)^2} \theta^4 + \cdots \right\} = \frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}} \sum_{\ell=0}^{\infty} \frac{P_{2\ell}^n(t)}{(4\pi t)^\ell} \theta^{2\ell}. \end{aligned} \tag{4.10}$$

Note in particular that from (4.8) and (4.9) the first polynomials P_0^n and P_2^n in the sequence are given respectively by

$$P_0^n(t) = 1 + \frac{1}{\Gamma\left(\frac{n}{2}\right)} \sum_{m=0}^{s-1} \mathsf{A}_m^n \Gamma\left(m + \frac{3}{2}\right) t^{s-m}, \tag{4.11}$$

$$\begin{aligned} P_2^n(t) = & -\pi \left\{ 1 + \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} \sum_{m=0}^{s-1} \mathsf{A}_m^n \Gamma\left(m + \frac{5}{2}\right) t^{s-m} \right. \\ & \left. - \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} \frac{(n-1)^2}{4} \sum_{m=0}^{s} \mathsf{A}_m^n \Gamma\left(m + \frac{3}{2}\right) t^{s-m+1} \right\}. \end{aligned} \tag{4.12}$$

Additionally in the case $n = 3$ the above reduces and results in the formulation

$$K_{\mathbb{S}^3}(t,\theta) \sim \frac{e^t}{(4\pi t)^{\frac{3}{2}}} \left\{ 1 - \frac{\theta^2 \pi}{(4\pi t)} \left(1 - \frac{2t}{3}\right) + \frac{\theta^4 \pi^2/2}{(4\pi t)^2} \left(1 - \frac{4t}{3} + \frac{28t^2}{45}\right) + \cdots \right\}. \tag{4.13}$$

(b) Even $n \geq 2$. Similar to what was done earlier in the case of odd n we use the polynomial representation of the multiplicity $M_k^n(\mathbb{S}^n)$ in the form[3]

$$\begin{aligned} M_k^n &= \frac{(k+n-2)!}{(n-1)!k!}(n+2k-1) = \frac{2k+n-1}{n-1} \prod_{j=1}^{n-2} \frac{k+j}{j} = \frac{2X_k}{(n-1)!} \prod_{j=\frac{1}{2}}^{s-\frac{1}{2}} (X_k^2 - j^2) \\ &= \frac{2X_k}{(n-1)!} \sum_{m=0}^{s} \mathsf{B}_m^n X_k^{2m} = \frac{2X_k \mathscr{B}(X_k)}{(n-1)!} \quad s = (n-2)/2, \end{aligned} \tag{4.14}$$

where $\mathscr{B} = \mathscr{B}_n$ is the degree $n-2$ polynomial as in (B.2) and $X_k = k + (n-1)/2$ with $k \geq 0$. Using the multiplicity (4.14) in the heat trace formula (1.10) and the root structure of the polynomial $\mathscr{B}(X)$ we obtain a more convenient form of the heat trace formula,

$$\begin{aligned} \Theta_{\mathbb{S}^n}(t) = \operatorname{tr} e^{-t\Delta_{\mathbb{S}^n}} &= 2 \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!} \sum_{k=0}^{\infty} X_k \mathscr{B}(X_k) e^{-X_k^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!} \sum_{m=0}^{s} \mathsf{B}_m^n \sum_{X_k=0}^{\infty} 2X_k^{2m+1} e^{-X_k^2 t} \\ &= \frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!} \sum_{m=0}^{s} (-1)^m \mathsf{B}_m^n \vartheta_2^{(m)}(t), \end{aligned} \tag{4.15}$$

where $\vartheta_2^{(m)}$ denote the respective derivatives of the Jacobi theta function ϑ_2 (see the Appendix at the end). This therefore gives for $\ell \geq 1$

[3] For $n = 2$ the argument largely simplifies as here $M_k^2 = 2k + 1$ and the product is trivial. We therefore assume hereafter that $n \geq 4$. For $n = 2$ see the remark at the end.

$$b_{2\ell}^{n}(t)=\frac{e^{\frac{(n-1)^2}{4}t}}{(n-1)!}\sum_{j=1}^{\ell}\frac{(-1)^{j}c_j^{\ell}(\alpha,\beta)}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n-1}{2}\right)^{2r}\sum_{m=0}^{s}(-1)^{m}\mathsf{B}_m^n\vartheta_2^{(m+j-r)}(t). \tag{4.16}$$

Theorem 4.2 (Even $n\geq 4$) *The spectral functions $b_{2\ell}^n=b_{2\ell}^n(t)$ associated with $\mathscr{X}=\mathbb{S}^n$ for $n\geq 4$ even admit the asymptotics as $t\searrow 0$*

$$b_{2\ell}^{n}(t)=\mathscr{R}_{\ell}\left(-\frac{d}{dt}\right)\operatorname{tr}e^{-t\Delta}\sim\omega_1^n\frac{e^{\frac{(n-1)^2 t}{4}}}{(4\pi t)^{\frac{n}{2}+\ell}}P_{2\ell}^{n}(t), \tag{4.17}$$

where $P_{2\ell}=P_{2\ell}(t)$ with $\ell\geq 0$ are the power series

$$P_0^n(t)=1+\frac{1}{s!}\left[\sum_{m=0}^{s-1}\mathsf{B}_m^n m!t^{s-m}+\sum_{m=0}^{s}(-1)^m\mathsf{B}_m^n\sum_{k=m}^{\infty}\frac{\mathtt{B}_k t^{s+k-m+1}}{(k-m)!}\right] \tag{4.18}$$

$$\begin{aligned}P_{2\ell}^{n}(t)=&\frac{4^{\ell}\pi^{\ell}}{s!}\sum_{j=1}^{\ell}\frac{c_j^{\ell}(\alpha,\beta)}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n-1}{2}\right)^{2r}\\&\times\sum_{m=0}^{s}\Big[(-1)^{r}\mathsf{B}_m^n(m+j-r)!t^{\ell+s-m-j+r}\\&+(-1)^{j+m}\mathsf{B}_m^n\sum_{k=m+j-r}^{\infty}\frac{\mathtt{B}_k t^{\ell+s+k-m-j+r+1}}{(k-m-j+r)!}\Big].\end{aligned} \tag{4.19}$$

Here s is as in (4.14), B_m^n are as in (B.2), $\mathtt{B}_j$ as in (A.3) and $c_j^{\ell}=c_j^{\ell}(\alpha,\beta)$ are the Jacobi coefficients with $\alpha=\beta=(n-2)/2$.

Inserting the asymptotics (4.17) with (4.18) and (4.19) in the Maclaurin expansion of the heat kernel we obtain the following description of the asymptotics of $K_{\mathbb{S}^n}(t,\theta)$:

$$\begin{aligned}K_{\mathbb{S}^n}(t,\theta)&=\frac{1}{\omega_1^n}\operatorname{tr}e^{-t\Delta}+\frac{1}{\omega_1^n}\sum_{\ell=1}^{\infty}\mathscr{R}_{\ell}\left(-\frac{d}{dt}\right)\operatorname{tr}e^{-t\Delta}\theta^{2\ell}\\&\sim\frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}}\left\{P_0^n(t)+\frac{P_2^n(t)}{(4\pi t)}\theta^2+\frac{P_2^n(t)}{(4\pi t)^2}\theta^4+\cdots\right\}=\frac{e^{\frac{(n-1)^2}{4}t}}{(4\pi t)^{\frac{n}{2}}}\sum_{\ell=0}^{\infty}\frac{P_{2\ell}^n(t)}{(4\pi t)^{\ell}}\theta^{2\ell}.\end{aligned} \tag{4.20}$$

Remark 4.3 Again it is seen from a similar set of calculations that in the case $n=2$ we have the formulation

$$
\begin{aligned}
K_{\mathbb{S}^2}(t,\theta) \sim{}& \frac{e^{\frac{t}{4}}}{4\pi t}\left[1+\sum_{j=0}^{\infty}\frac{\mathsf{B}_j}{j!}t^{j+1}\right] \\
&-\frac{\theta^2\pi e^{\frac{t}{4}}}{(4\pi t)^2}\left[1-\frac{t}{4}-\frac{t^2}{4\cdot 6\cdot 2}-\sum_{j=1}^{\infty}\frac{\mathsf{B}_j}{(j-1)!}\left(t^{j+1}+\frac{t^{j+2}}{4j}\right)\right] \\
&+\frac{\theta^4(\pi^2/2)e^{\frac{t}{4}}}{(4\pi t)^3}\left[1-\frac{7t}{12}+\frac{11t^2}{96}+\frac{13t^3}{7!}+\frac{7\cdot 11t^4}{96\cdot 8\cdot 5!}+\right. \\
&\left.+\frac{1}{2}\sum_{j=2}^{\infty}\frac{\mathsf{B}_j}{(j-2)!}\left(t^{j+1}+\frac{7t^{j+2}}{6(j-1)}+\frac{11t^{j+3}}{48j(j-1)}\right)\right]+O(\theta^6).
\end{aligned}
\tag{4.21}
$$

The Complex Projective Space $\mathbf{P}^n(\mathbb{C})$ We now turn to the complex projective space $\mathbf{P}^n(\mathbb{C})$ (of real dimension $2n$) and note that

$$
\begin{aligned}
K_{\mathbf{P}^n(\mathbb{C})}(t,\theta) &= \sum_{\ell=0}^{\infty}\frac{b^n_{2\ell}(t)}{\omega_3^n}\theta^{2\ell} = \frac{1}{\omega_3^n}\operatorname{tr} e^{-t\Delta}+\frac{1}{\omega_3^n}\sum_{\ell=1}^{\infty}\mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta}\theta^{2\ell} \\
&= \mathbf{I}+\theta^2\mathbf{II}+\theta^4\mathbf{III}+\cdots,
\end{aligned}
\tag{4.22}
$$

where using the associated Jacobi coefficients $c_j^\ell(\alpha,\beta)$ (with $\alpha = n-1$ and $\beta = 0$) we have

$$
\mathbf{I} = \frac{1}{\omega_3^n}\operatorname{tr} e^{-t\Delta}, \quad \mathbf{II} = \frac{1/2n}{2!\omega_3^n}\frac{d}{dt}\operatorname{tr} e^{-t\Delta},
$$

$$
\mathbf{III} = \frac{1/4n}{4!\omega_3^n}\left(\frac{d}{dt}+\frac{3}{n+1}\frac{d^2}{dt^2}\right)\operatorname{tr} e^{-t\Delta}.
$$

(a) Odd $n \geq 3$. Here we write the multiplicity function $M_k^n(\mathbf{P}^n(\mathbb{C}))$ in the product form[4]

$$
\begin{aligned}
M_k^n &= \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 = \frac{2Y_k}{n!(n-1)!}\prod_{j=\frac{1}{2}}^{s+\frac{1}{2}}(Y_k^2-j^2)^2 \\
&= \frac{2Y_k}{n!(n-1)!}\sum_{m=0}^{2s+2}\mathsf{C}_m^n Y_k^{2m} = \frac{2Y_k\mathscr{C}(Y_k)}{n!(n-1)!}, \quad s=(n-3)/2
\end{aligned}
\tag{4.23}
$$

where $\mathscr{C} = \mathscr{C}_n$ is the degree $2n-2$ polynomial as in (B.3) and $Y_k = k+n/2$ with $k \geq 0$. Now by inserting the multiplicity (4.23) into the heat trace formula (2.2) we have after some straightforward calculation as before

[4] For the case $n = 1$ see the remark at the end.

$$\begin{aligned}\Theta_{\mathbf{P}^n(\mathbb{C})} &= \operatorname{tr} e^{-t\Delta_{\mathbf{P}^n(\mathbb{C})}} = \frac{2e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{k=0}^{\infty} Y_k \mathscr{C}(Y_k) e^{-Y_k^2 t} \\ &= \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{m=0}^{n-1} \mathsf{C}_m^n \sum_{Y_k=0}^{\infty} 2Y_k^{2m+1} e^{-Y_k^2 t} \\ &= \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{m=0}^{n-1} (-1)^m \mathsf{C}_m^n \vartheta_2^{(m)}(t)\end{aligned} \tag{4.24}$$

(see the Appendix for the notation on polynomials and the coefficients). Using again the Leibniz rule we now write the Maclaurin heat coefficients $b_{2m}^n = b_{2m}^n(t)$ in terms of the theta function ϑ_2, namely,

$$b_{2\ell}^n(t) = \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{j=1}^{\ell}(-1)^j \frac{c_j^\ell(\alpha,\beta)}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n^2}{4}\right)^r \sum_{m=0}^{n-1}(-1)^m \mathsf{C}_m^n \vartheta_2^{(m+j-r)}(t). \tag{4.25}$$

Theorem 4.4 (Odd $n \geq 3$) *The spectral functions $b_{2\ell}^n = b_{2\ell}^n(t)$ associated with $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$ for $n \geq 3$ odd admit the asymptotics as $t \searrow 0$*

$$b_{2\ell}^n(t) = \mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta} \sim \omega_3^n \frac{e^{\frac{n^2}{4}t}}{(4\pi t)^{n+\ell}} Q_{2\ell}^n(t), \tag{4.26}$$

where $Q_{2\ell} = Q_{2\ell}(t)$ with $\ell \geq 0$ are the power series

$$Q_0^n(t) = 1 + \sum_{m=0}^{n-2}\frac{\mathsf{C}_m^n m!}{(n-1)!}t^{n-m-1} + \sum_{m=0}^{n-1}\frac{(-1)^m \mathsf{C}_m^n}{(n-1)!}\sum_{k=m}^{\infty}\frac{\mathsf{B}_k t^{n+k-m}}{(k-m)!} \tag{4.27}$$

$$\begin{aligned}Q_{2\ell}^n(t) = {} & \frac{4^\ell \pi^\ell}{(n-1)!}\sum_{j=1}^{\ell}\frac{c_j^\ell}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n}{2}\right)^{2r} \\ & \times \sum_{m=0}^{n-1}\Big[(-1)^r \mathsf{C}_m^n (m+j-r)! t^{\ell+n-m-j+r-1} \\ & + (-1)^{j+m}\mathsf{C}_m^n \sum_{k=m+j-r}^{\infty}\frac{\mathsf{B}_k t^{\ell+n+k-m-j+r}}{(k-m-j+r)!}\Big]\end{aligned} \tag{4.28}$$

where C_m^n are as in (B.3), B_j as in (A.3) and $c_j^\ell = c_j^\ell(\alpha,\beta)$ are the Jacobi coefficients with $\alpha = n-1$, $\beta = 0$.

Inserting the asymptotics (4.26) with (4.27) and (4.28) into the Macluarin expansion of the heat kernel we obtain the following formulation of the asymptotics of $K_{\mathbf{P}^n(\mathbb{C})}(t,\theta)$:

$$
\begin{aligned}
K_{\mathbf{P}^n(\mathbb{C})}(t,\theta) &= \frac{1}{\omega_3^n}\operatorname{tr} e^{-t\Delta} + \frac{1}{\omega_3^n}\sum_{\ell=1}^{\infty}\mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta}\theta^{2\ell} \\
&\sim \frac{e^{\frac{n^2}{4}t}}{(4\pi t)^n}\left\{Q_0(t) + \frac{Q_2(t)}{(4\pi t)}\theta^2 + \frac{Q_4^n(t)}{(4\pi t)^2}\theta^4 + \cdots\right\} = \frac{e^{\frac{n^2}{4}t}}{(4\pi t)^n}\sum_{\ell=0}^{\infty}\frac{Q_{2\ell}^n(t)}{(4\pi t)^\ell}\theta^{2\ell}.
\end{aligned}
\tag{4.29}
$$

Remark 4.5 A similar reasoning gives for $n = 1$ the formulation (compare with $\mathscr{X} = \mathbb{S}^2$)

$$
\begin{aligned}
K_{\mathbf{P}^1(\mathbb{C})}(t,\theta) \sim \frac{e^{\frac{t}{4}}}{4\pi t}\Bigg\{1 + t\sum_{k=0}^{\infty}\frac{\mathsf{B}_k t^k}{k!} - \frac{\theta^2\pi}{(4\pi t)}\left(1 - \frac{t}{4} + \cdots\right) \\
+ \frac{\theta^4\pi^2/2}{(4\pi t)^2}\left(1 - \frac{7t}{12} + \cdots\right) + O(\theta^6)\Bigg\}.
\end{aligned}
\tag{4.30}
$$

(b) Even $n \geq 2$. We start by writing the multiplicity function $M_k^n(\mathbf{P}^n(\mathbb{C}))$ in a polynomial form[5]

$$
\begin{aligned}
M_k^n &= \frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2 = \frac{2Y_k^3}{n!(n-1)!}\prod_{j=1}^{s}\left(Y_k^2 - j^2\right)^2 \\
&= \frac{2Y_k^3}{n!(n-1)!}\sum_{m=0}^{2s}\mathsf{D}_m^n\left(k + \frac{n}{2}\right)^{2m} = \frac{2Y_k^3\mathscr{D}(Y_k)}{n!(n-1)!}, \qquad s = (n-2)/2
\end{aligned}
\tag{4.31}
$$

where $\mathscr{D} = \mathscr{D}_n$ is the degree $2n-4$ polynomial as in (B.4) and $Y_k = k + n/2$ with $k \geq 0$ (see the Appendix for the notation on the coefficients). If we now insert the multiplicity (4.31) into the heat trace formula (2.2) we obtain

$$
\begin{aligned}
\Theta_{\mathbf{P}^n(\mathbb{C})} = \operatorname{tr} e^{-t\Delta_{\mathbf{P}^n(\mathbb{C})}} &= \frac{2e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{k=0}^{\infty}Y_k^3\mathscr{D}(Y_k)e^{-Y_k^2 t} \\
&= \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{m=0}^{n-2}\mathsf{D}_m^n\sum_{k=0}^{\infty}2Y_k^{2m+3}e^{-Y_k^2 t} \\
&= \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{m=0}^{n-2}(-1)^{m+1}\mathsf{D}_m^n\vartheta_3^{(m+1)}(t).
\end{aligned}
\tag{4.32}
$$

By taking advantage of the Leibniz rule we can now write the spectral functions $b_{2m}^n = b_{2m}^n(t)$ in terms of the Jacobi function ϑ_3 as

[5] For $n = 2$ the argument largely simplifies as here $M_k^2 = (k+1)^3$ and the product is trivial. We henceforth assume that $n \geq 4$. For $n = 2$ see the remark at the end.

$$b_{2\ell}^n(t) = \frac{e^{\frac{n^2}{4}t}}{n!(n-1)!}\sum_{j=1}^{\ell}(-1)^j\frac{c_j^\ell(\alpha,\beta)}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n^2}{4}\right)^r\sum_{m=0}^{n-2}(-1)^{m+1}\mathsf{D}_m^n\vartheta_3^{(m+j-r+1)}(t). \tag{4.33}$$

Theorem 4.6 (Even $n \geq 4$) *The spectral functions $b_{2\ell}^n = b_{2\ell}^n(t)$ associated with $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$ for $n \geq 4$ even admit the asymptotics as $t \searrow 0$*

$$b_{2\ell}^n(t) = \mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta} \sim \omega_3^n\frac{e^{\frac{n^2}{4}t}}{(4\pi t)^{n+\ell}}Q_{2\ell}^n(t), \tag{4.34}$$

where $Q_{2\ell} = Q_{2\ell}(t)$ with $\ell \geq 0$ are the power series

$$Q_0^n(t) = 1 + \sum_{m=0}^{n-3}\frac{\mathsf{D}_m^n(m+1)!}{(n-1)!}t^{n-m-2} + \sum_{m=0}^{n-2}\frac{(-1)^{m+1}\mathsf{D}_m^n}{(n-1)!}\sum_{k=m+1}^{\infty}\frac{\mathsf{B}_k t^{n+k-m-1}}{\Gamma(k-m)} \tag{4.35}$$

$$\begin{aligned}Q_{2\ell}^n(t) = {} & \frac{4^\ell\pi^\ell}{(n-1)!}\sum_{j=1}^{\ell}\frac{c_j^\ell}{(2\ell)!}\sum_{r=0}^{j}\binom{j}{r}\left(\frac{n}{2}\right)^{2r}\sum_{m=0}^{n-2}\Big[(-1)^r\mathsf{D}_m^n(m+j-r+1)! \\ & \times t^{\ell+n-m-j+r-2} + (-1)^{j+m+1}\mathsf{D}_m^n\sum_{k=m+j-r+1}^{\infty}\frac{\mathsf{C}_k t^{\ell+n+k-m-j+r-1}}{\Gamma(k-m-j+r)}\Big]\end{aligned} \tag{4.36}$$

where D_m^n are as in (B.4), C_j as in (A.3) and $c_j^\ell = c_j^\ell(\alpha,\beta)$ are the Jacobi coefficients with $\alpha = n-1$, $\beta = 0$.

Inserting the asymptotics (4.34) with (4.35) and (4.36) into (4.22) we obtain the following alternative description of the heat kernel $K_{\mathbf{P}^n(\mathbb{C})}(t,\theta)$:

$$\begin{aligned}K_{\mathbf{P}^n(\mathbb{C})}(t,\theta) &= \frac{1}{\omega_3^n}\operatorname{tr} e^{-t\Delta} + \frac{1}{\omega_3^n}\sum_{\ell=1}^{\infty}\mathscr{R}_\ell\left(-\frac{d}{dt}\right)\operatorname{tr} e^{-t\Delta}\theta^{2\ell} \\ &\sim \frac{e^{\frac{n^2}{4}t}}{(4\pi t)^n}\left\{Q_0^n(t) + \frac{Q_2^n(t)}{(4\pi t)}\theta^2 + \frac{Q_4^n(t)}{(4\pi t)^2}\theta^4 + \cdots\right\} = \frac{e^{\frac{n^2}{4}t}}{(4\pi t)^n}\sum_{\ell=0}^{\infty}\frac{Q_{2\ell}^n(t)}{(4\pi t)^\ell}\theta^{2\ell}.\end{aligned} \tag{4.37}$$

Remark 4.7 Using a similar reasoning in the special case $n = 2$ we obtain the formulation

$$\begin{aligned}K_{\mathbf{P}^2(\mathbb{C})}(t,\theta) \sim \frac{e^t}{(4\pi t)^2}\Big\{ & 1 - t\sum_{k=1}^{\infty}\frac{\mathsf{C}_k t^k}{(k-1)!} - \frac{\theta^2\pi}{(4\pi t)}\left(1 - \frac{t}{2} + \cdots\right) \\ & + \frac{\theta^4\pi^2/2}{(4\pi t)^2}(1 - t + \cdots) + O(\theta^6)\Big\}.\end{aligned} \tag{4.38}$$

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Appendix A: The theta functions $\vartheta_1, \vartheta_2, \vartheta_3$ and their asymptotics

In this Appendix we gather together the definitions and asymptotics of the Jacobi theta functions ϑ_1, ϑ_2 and ϑ_3 for $t > 0$ and as $t \searrow 0$ referred to earlier in Sect. 4. First we have

$$\vartheta_1(t) = \sum_{j=-\infty}^{+\infty} e^{-j^2 t} = \sqrt{\pi/t} \sum_{j=-\infty}^{+\infty} e^{-\pi^2 j^2/t} \sim \sqrt{\pi/t}. \tag{A.1}$$

$$\vartheta_1^{(m)}(t) \sim (-1)^m \frac{(2m-1)!!}{2^m} \sqrt{\pi/t^{2m+1}}, \quad m \geq 1. \tag{A.2}$$

The second identity on the first line (A.1) follows from the Poisson summation formula and the second line follows by successive differentiation and taking the asymptotics as $t \searrow 0$. To proceed further we introduce the scalars

$$\mathtt{B}_j = \frac{(-1)^j}{j+1}\left(1 - 2^{-2j-1}\right) B_{2j+2}, \quad \mathtt{C}_j = \frac{(-1)^j}{j+1} B_{2j+2}, \quad j \geq 0, \tag{A.3}$$

where B_{2j}s are the well-known Bernoulli numbers (see e.g., [15, 26]). Now we have

$$\vartheta_2(t) = \sum_{j=0}^{\infty} (2j+1) e^{-\left(j+\frac{1}{2}\right)^2 t} = \frac{1}{t} + \sum_{j=0}^{n} \frac{\mathtt{B}_j t^j}{j!} + O(t^{n+1}), \tag{A.4}$$

$$\vartheta_2^{(m)}(t) = \frac{(-1)^m m!}{t^{1+m}} + \sum_{j=m}^{n} \frac{\mathtt{B}_j t^{j-m}}{(j-m)!} + O(t^{n+1}), \quad m \geq 1, \tag{A.5}$$

and likewise

$$\vartheta_3(t) = 2 \sum_{j=1}^{\infty} j e^{-j^2 t} = \frac{1}{t} + \sum_{j=0}^{n} \frac{\mathtt{C}_j t^j}{j!} + O(t^{n+1}) \tag{A.6}$$

$$\vartheta_3^{(m)}(t) \sim \frac{(-1)^m m!}{t^{1+m}} + \sum_{j=m}^{n} \frac{\mathtt{C}_j t^{j-m}}{(j-m)!} + O(t^{n+1}), \quad m \geq 1. \tag{A.7}$$

Note that in the asymptotics in (A.1), (A.2) the remainder term is exponentially small, that is, $O(e^{-1/t})$ whereas in the other cases as indicated the remainder between the series on the left and the nth partial sum on the right is $O(t^{n+1})$. Also upon referring to Table 2 it is easily seen that $\vartheta_1 = \Theta_{\mathbb{S}^1}(t)$ while $\vartheta_2 = e^{-t/4}\Theta_{\mathbb{S}^2}(t)$ and $d\vartheta_3/dt = -2e^{-t}\Theta_{\mathbf{P}^2(\mathbb{C})}$.

Appendix B: The polynomials $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ and the associated coefficients

Below n is a positive integer: $n \geq 3$ odd in (B.1), (B.3) and $n \geq 4$ even in (B.2), (B.4). Likewise s is a non-negative integer introduced purely for notational convenience in line with that of Sect. 4. Note that in the second and third lines j runs through half-integers (non-integers) while in the first and fourth j runs through integers and in either case between the upper and lower bounds of the respective products.

$$\mathscr{A}(X) = \prod_{j=0}^{s}(X^2 - j^2) = \sum_{m=0}^{s} \mathsf{A}_m^n X^{2m+2} \quad s = (n-3)/2 \tag{B.1}$$

$$\mathscr{B}(X) = \prod_{j=\frac{1}{2}}^{s-\frac{1}{2}}(X^2 - j^2) = \sum_{m=0}^{s} \mathsf{B}_m^n X^{2m} \quad s = (n-2)/2 \tag{B.2}$$

$$\mathscr{C}(X) = \prod_{j=\frac{1}{2}}^{s+\frac{1}{2}}(X^2 - j^2)^2 = \sum_{m=0}^{2s+2} \mathsf{C}_m^n X^{2m} \quad s = (n-3)/2 \tag{B.3}$$

$$\mathscr{D}(X) = \prod_{j=1}^{s}(X^2 - j^2)^2 = \sum_{m=0}^{2s} \mathsf{D}_m^n X^{2m} \quad s = (n-2)/2 \tag{B.4}$$

These polynomials are utilised in the representation of the multiplicity functions $M_k^n(\mathscr{X})$ for spheres and projective spaces in particular.

Appendix C: Calculations of the first few Jacobi coefficients $c_j^\ell(\alpha, \beta)$ with $(j, \ell \geq 1, j \leq \ell)$

Let $y = y(t)$ denote the normalised Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)} = P_k^{(\alpha,\beta)}(t)/P_k^{(\alpha,\beta)}(1)$. Then it follows by straightforward differentiation that

$$\frac{d^2}{d\theta^2} y(\cos\theta) = y''(\cos\theta)\sin^2\theta - y'(\cos\theta)\cos\theta, \tag{C.1}$$

which, by comparing with the Jacobi differential equation (1.16), gives

$$\left.\frac{d^2}{d\theta^2} y(\cos\theta)\right|_{\theta=0} = -\frac{k(k+\alpha+\beta+1)}{2(\alpha+1)} \Rightarrow c_1^1 = -\frac{1}{2(\alpha+1)}. \tag{C.2}$$

In the same way we have

$$\left.\frac{d^4}{d\theta^4} y(\cos\theta)\right|_{\theta=0} = \left\{c_1^2[k(k+\alpha+\beta+1)] + c_2^2[k(k+\alpha+\beta+1)]^2\right\}, \tag{C.3}$$

where

$$c_1^2 = -\frac{(\alpha+3\beta+2)}{4(\alpha+1)(\alpha+2)}, \quad c_2^2 = \frac{3}{4(\alpha+1)(\alpha+2)}. \tag{C.4}$$

The sixth derivative can be calculated in much the same way and it is given by

$$\left.\frac{d^6 y(\cos\theta)}{d\theta^6}\right|_{\theta=0} = \left\{c_1^3[k(k+\alpha+\beta+1)] + c_2^3[k(k+\alpha+\beta+1)]^2 + c_3^3[k(k+\alpha+\beta+1)]^3\right\}, \tag{C.5}$$

where

$$c_1^3 = -\frac{4(\alpha^2+5\alpha+6)+30\beta(\beta+\alpha+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)}, \tag{C.6}$$

$$c_2^3 = \frac{15\alpha+45\beta+30}{8(\alpha+1)(\alpha+2)(\alpha+3)}, \tag{C.7}$$

$$c_3^3 = -\frac{15}{8(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{C.8}$$

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